**Chapter 2: Functions, Limits and Continuity**

Table of Contents

[2.1 Variables and Functions 3](#_Toc82187673)

[2.2 Single and Multiple-Valued Functions 4](#_Toc82187674)

[2.3 Inverse Functions 5](#_Toc82187675)

[2.4 Transformations 6](#_Toc82187676)

[2.5 Curvilinear Coordinates 7](#_Toc82187677)

[2.6 The Elementary Functions 8](#_Toc82187678)

[Polynomial Functions 8](#_Toc82187679)

[Rational Algebraic Functions 8](#_Toc82187680)

[Exponential Functions 9](#_Toc82187681)

[Trigonometric Functions 9](#_Toc82187682)

[Hyperbolic Functions 10](#_Toc82187683)

[Logarithmic Functions 11](#_Toc82187684)

[Inverse Trigonometric Functions 12](#_Toc82187685)

[Inverse Hyperbolic Functions 13](#_Toc82187686)

[2.9 Limits 14](#_Toc82187687)

[2.10 Theorems on Limits 16](#_Toc82187688)

[2.12 Continuity 17](#_Toc82187689)

[2.13 Theorems on Continuity 18](#_Toc82187690)

## 2.1 Variables and Functions

If each value of a complex variable corresponds to one or more values of another complex variable , we can say that is a **function** of , i.e. .

Thus, if we give a value to , then we can find a value for . For example, if ,

## 2.2 Single and Multiple-Valued Functions

If every value of corresponds to a single value of , then we say that is a **single-valued function** of . If each value of corresponds to more than one value of , we say that is a **multiple-valued function** of .

A multiple-valued function can be considered to be a **collection** of single-valued functions, each member being a **branch** of the function. One of the branches is considered to be the **principal branch**, and the corresponding value the **principal value**.

For example, is a single-valued function, but is a multiple-valued function.

Unless otherwise stated, we should always assume that a function is single-valued.

## 2.3 Inverse Functions

If is a function, is the **inverse** of that function.

## 2.4 Transformations

If is a single-valued function of , we can say that . Thus, given a point on the plane, a corresponding point can be found on the plane. We can say that the first point was **transformed** onto the second.

## 2.5 Curvilinear Coordinates

Given the transformation , we call from the plane the **rectangular coordinates** corresponding to the point and we call from the plane the **curvilinear coordinates** of .

## 2.6 The Elementary Functions

### Polynomial Functions

Polynomial functions are defined as

where , are **complex constants** and is a **non-negative integer** called the **degree** of the polynomial .

The transformation is called a **linear transformation**.

### Rational Algebraic Functions

These are defined as

where both and are **polynomial functions**.

We sometimes call these functions **rational transformations**.

The special case where and is often called a **bilinear** or **fractional linear function**.

### Exponential Functions

If , we can say that .

Recalling Euler’s formula, this can also be written as

Here, is the **natural base of logarithms**.

If is real and positive, we define , where is the natural logarithm of .

### Trigonometric Functions

Using Euler’s formula, we can define trigonometric functions in terms of exponential functions.

Simultaneously solving these equations, we can define

If we replace with a complex function , we can get **complex trigonometric functions**.

All trigonometric properties also apply.

### Hyperbolic Functions

A trigonometric function always defines a circle. For a particular value of , and , which gives a point . For the variables and , is always true.

In a hyperbolic function on the other hand, the equation is always true. In this case, and .

The relationship between complex trigonometric and hyperbolic functions can be set up as follows:

From these relationships, it is possible to derive the following hyperbolic functions for complex hyperbolic functions:

The following properties can also be derived:

### Logarithmic Functions

If , then . Thus, the natural logarithmic function is the **inverse** of the exponential function. It can be defined as:

where .

is multi-branched, and the principal value occurs when in this case.

The logarithmic function can also be defined for real bases other than . Thus, if , .

### Inverse Trigonometric Functions

If , then and is called the **inverse** sine of or the **arc** sine of . Similarly, all inverse trigonometric complex functions can be defined in terms of natural logarithms.

Let .

Let .

Solving this, we get

Above, we used instead of , since , which cannot be negative.

Following a similar process, we can get all the complex inverse trigonometric functions.

### Inverse Hyperbolic Functions

Similar to how we derived the equations for complex inverse trigonometric functions, we can also derive the equations for complex inverse hyperbolic functions.

## 2.9 Limits

For a **real-valued function**, if the function is undefined at and , where , i.e. the left-hand limit and the right-hand limit are the same, a **limiting** value, , is said to exist.

Note that the **limit** is not the same as the **actual value**. We know that the actual value is undefined in this case.

**Complex-valued** functions can have similar limits, they are just thought of a little differently, since , meaning we have **two** independent variables.

For real-valued functions, simply means that the point on a **straight line** approaches . For complex-valued functions, means that the point , which is on a **graph**, approaches . It may be approaching this along a straight line or a curved line, depending on the function . Since it is on a graph, there are multiple possible ways in which could approach . If the limits for the approaches from all possible directions is equal, then a **limiting value** is said to exist. This limiting value will be **unique**.

Mathematically though, we can write it just the same. For example, say a function, , is undefined at . Then we can say that the limiting value at is .

Geometrically speaking, if the limit for a complex function is at the point , it means the difference in the **absolute value** between and is **minimal**. This is the  **definition**, since we are basically proving that if . Additionally, must be dependent on .

Going back to the point about being able to approach from many different directions, how do we prove that a limit exists? After all, we cannot check for an infinite number of paths. Instead, we try to prove the opposite, that the values obtained by approaching from just two specific directions are **not the same**, meaning a limit **cannot exist**.

## 2.10 Theorems on Limits

Suppose and . Then,

1. if

## 2.12 Continuity

For **real-valued functions**, the function is said to be **continuous** when the limiting value and the functional value are the same, i.e. .

Similarly, for **complex-valued functions**, the function is said to be **continuous** when the limiting value and the functional value are the same, i.e. .

Continuity is actually dependent on **three conditions**:

1. exists
2. must be defined

Points at which the function fails to be continuous are called **discontinuities** of .

If the limiting value exists but is not equal to the functional value, it is called a **removable discontinuity**, since if we redefine as the limiting value, the function becomes continuous.

If is continuous for all points in a region, it is said to be **continuous in the region**.

## 2.13 Theorems on Continuity

1. If and are **continuous**, then so are , , and . In the last case, .
2. If a function is continuous within a **finite region**, then the function is either

* Polynomial
* or

1. Suppose is continuous at and is continuous at . If , then the **composite function** is continuous at .

A **continuous function** of a **continuous function** is **continuous**.

1. If is continuous in a **closed** and **bounded** region, then it is **bounded in the region**, i.e. there is a constant , such that for all points in the region.
2. If is **continuous** in a region, then the real and imaginary parts of are also continuous in that region.